ISRAEL JOURNAL OF MATHEMATICS, Vol. 52, Nos. 1-2, 1985

TWO REMARKS ON EQUIVELAR MANIFOLDS

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ABSTRACT

It has been pointed out that a construction for equivelar polyhedral manifolds described in an earlier paper by the present authors may possibly not be performed; an alternative construction is given here. Further, it is shown that certain equivelar polyhedra cannot have equiangular (and hence regular) faces.

1. Introduction

We recall from [4] that an equivelar polyhedron of type $\{p,q\}$ or in the class $\mathcal{M}_{p,q}$ is a closed polyhedral 2-manifold embedded in $\mathbf{E}^3$, whose faces are all convex p-gons and whose vertices are all q-valent. In [4], we showed that certain of the classes $\mathcal{M}_{p,q}$ realized all but finitely many genera, while in [5], we proved that each of the classes $\mathcal{M}_{3,q}$ $(q \geqq 7)$, $\mathcal{M}_{4,q}$ $(q \geqq 5)$ and $\mathcal{M}_{p,4}$ $(p \geqq 5)$ contains infinitely many combinatorial types.

However, it has been pointed out by Goossens [2] that one of the constructions we employed in [4], namely Method B, does not work, at least in the generality in which we stated it. This unfortunately invalidates Theorem 2 (d) (for the types $\{3,9;g\}$ and $\{4,6;g\}$; g denotes the genus), and also eliminates the types $\{3,12;g\}$ and $\{4,8;g\}$ of (e) below. Now the last two are less important, since only the sparse sequence of genera $g = 66n + 73$ $(n \geqq 0)$ was obtained; the results of [5] also give infinite (though different) sequences of genera for these types.

For special sequences of genera of $\{4,6;g\}$ there are variants of Method B which lead to equivelar manifolds with additional symmetry properties. For example in [6] there is a simple construction which leads to the sequence $\{4,6;g\}$, $g = 3n$, $n \geqq 2$ with tetrahedral symmetry group. With a similar construction

Received November 26, 1984

$\{4,6;g\}$, $g=3n+1$, $n \geqq 3$ is obtained with the dihedral symmetry group. But these methods do not work for all genera, so in the first part of this note we shall give an alternative construction which will restore the missing parts of Theorem 2 (d). Rather than dealing with 4-polytopes, we shall work instead with their Schlegel diagrams.

The second part concerns the non-existence of equivelar manifolds of type $\{p,4\}$ ($p \geqq 4$) whose faces are equiangular; this therefore excludes the possibility of all the faces being regular. (This result is in a similar spirit to that of [1], although the latter is combinatorial in nature.)

2. The classes $\mathcal{M}_{3,9}$ and $\mathcal{M}_{4,6}$

For the reader's convenience, we repeat the statement of the appropriate parts of Theorem 2 of [4].

THEOREM 1. *There exist equivelar polyhedra of type* $\{3,9;g\}$ *and* $\{4,6;g\}$, *for* $g=6,9,10$ *and* $g \geqq 12$.

Only the class $\mathcal{M}_{4,6}$ is really of interest here, since our new construction will amount combinatorially speaking to the same as that obtained by Method B, and so the same modification can be applied to $\{4,6;g\}$ to obtain $\{3,9;g\}$.

Let $\mathcal{D}$ be a Schlegel diagram in $\mathbf{E}^3$ of a simplicial 4-polytope (or, rather, a 3-diagram isomorphic to such a Schlegel diagram — see [3] for further details). Let F_0 be the base of $\mathcal{D}$, and let the remaining 3-cells be $F_1,\dots,F_n$, with $F_1,\dots,F_4$ having common 2-cells with F_0. We denote the 2-cells of $\mathcal{D}$ by $F_{ij}=F_i \cap F_j$ (and only use this notation if F_i and F_j share a 2-cell).

Suppose that there are points $p_i \in \operatorname{int} F_i$ ($i=1,\dots,n$), such that, for any pair F_i, F_j of adjacent 3-cells,

$$[p_i,p_j] \cap \operatorname{relint} F_{ij} \neq \emptyset.$$

(Here, $[a,b]$ denotes the line segment joining a and b.) With some λ satisfying $0<\lambda<1$, let H_i be the homothetic copy of F_i:

$$H_i=(1-\lambda)p_i+\lambda F_i \qquad (i=1,\dots,n);$$

moreover, write

$$H_{ij}=(1-\lambda)p_i+\lambda F_{ij}$$

for its face corresponding to F_{ij}, and

$$K_{ij}=\operatorname{conv}(H_{ij} \cup H_{ji}).$$

If λ is chosen sufficiently small, the prism K_{ij} meets the triangle F_{ij} only in its relative interior, and so the only intersections between the K_{ij}'s are of the kind

$$K_{ij} \cap K_{ik} = H_{ij} \cap H_{ik}.$$

This remains true for all i and j, if we define

$$K_{0j} = \mathrm{conv}(F_{0j} \cup H_{j0}) \qquad (j = 1, \ldots, 4).$$

Thus the quadrilateral faces of the K_{ij} form a polyhedral manifold of type $\{4,6; n+2\}$; the calculation of the genus $g = n+2$ is exactly as in [4].

It remains for us to show the existence of simplicial 3-diagrams $\mathcal{D}$ with $n+1 = 5,8,9$ and $n+1 \geqq 11$ 3-cells, which admit a choice of appropriate points $p_1, \ldots, p_n$ as above; let us call such a diagram *suitable*. There are two steps in the proof: we must find initial examples, and then apply an inductive procedure.

We describe the inductive procedure first, because this motivates our search for initial examples. If $\mathcal{D}$ is a suitable diagram, and $p_i \in \mathrm{int}\, F_i$ one of the chosen points, then it is clear that any other point p'_i in a sufficiently small neighbourhood of p_i may be chosen instead of p_i. Now, if we take the stellar subdivision of $\mathcal{D}$ at p_i, we replace the 3-cell F_i by four 3-cells, $F_i^1, \ldots, F_i^4$, say. We can obviously choose $p_i^j \in \mathrm{relint}\, F_i^j$ $(j = 1, \ldots, 4)$, sufficiently near p_i, which will satisfy our required conditions. So, we have obtained a new suitable diagram with three more 3-cells than $\mathcal{D}$. Hence, we need only find suitable diagrams $\mathcal{D}$ with $n+1 = 5,9$ and 13 3-cells.

The first case $n+1=5$ is trivial: it is just the stellar subdivision of a tetrahedron at an interior point.

For $n+1=9$, we take a diagram isomorphic to a Schlegel diagram of the cyclic polytope $C(6,4)$ (see [3]). If we choose it appropriately, we shall see that we can derive the case $n+1=13$ from it. The base F_0 of $\mathcal{D}$ is the tetrahedron with vertices $a = (2,2,2)$, $b = (-2,-2,2)$, $c = (2,-2,-2)$ and $d = (-2,2,-2)$, and the two remaining vertices are $e = (0,0,1)$ and $f = (0,0,-1)$. The 8 other 3-cells have vertices

F_1: *abce*,	F_5: *acef*,
F_2: *abde*,	F_6: *adef*,
F_3: *acdf*,	F_7: *bcef*,
F_4: *bcdf*,	F_8: *bdef*.

The points $p_1, \ldots, p_8$, given by

$$
\begin{aligned}
p_1 &= (\alpha, -\alpha, \beta), & p_5 &= (\alpha, 0, 0),\\
p_2 &= (-\alpha, \alpha, \beta), & p_6 &= (0, \alpha, 0),\\
p_3 &= (\alpha, \alpha, -\beta), & p_7 &= (0, -\alpha, 0),\\
p_4 &= (-\alpha, -\alpha, -\beta), & p_8 &= (-\alpha, 0, 0),
\end{aligned}
$$

with $\alpha = \frac{1}{5}$ and $\beta = \frac{3}{2}$, can be checked to yield a suitable diagram. Obviously there is some freedom in choosing the p_i. (The symmetry of the figure makes this clear.)

For the final case $n + 1 = 13$, we take the stellar subdivision of $\mathscr{D}$ at the point $g = (0,0,0)$ of the edge $[e,f]$. Each of the 3-cells F_j $(j = 5,\ldots,8)$ is replaced by two 3-cells F_j^+ and F_j^-, and the corresponding point p_j is split into two points $p_j^\pm \in \operatorname{relint} F_j^\pm$, as, for example,

$$
\begin{aligned}
F_5^+ &: aceg, & p_5^+ &= (\alpha, 0, \tfrac{1}{2}),\\
F_5^- &: acfg, & p_5^- &= (\alpha, 0, -\tfrac{1}{2}),
\end{aligned}
$$

and so on. Again, it may easily be checked that this diagram is suitable.

3. Equivelar polyhedra with equiangular faces

In calling an equivelar polyhedron of type $\{p,q\}$ *equiangular*, we mean that each of its 2-faces is an equiangular polygon. This implies, of course, that the angle of each polygonal face is $\pi(1-2/p)$. In this section, we shall prove

THEOREM 2. *There is no equiangular equivelar polyhedron of type* $\{p,4\}$ *for* $p \geqq 4$.

As a consequence (as we mentioned in the introduction), no such polyhedral manifold can have regular faces.

To prove the theorem, let v be any vertex of such an equiangular equivelar polyhedron M. If S is a sufficiently small sphere with centre v, then $S \cap M$ is a spherical 4-gon P; its 4 edges have the same length, and so P must be a rhombus. Supposing S to be the unit sphere (as we clearly may), it follows that the diagonals of P lie in perpendicular planes through $v = 0$, and so we can choose coordinates so that the vertices of P are

$$
\begin{aligned}
a_\pm &= (0, \pm\sin\alpha, \cos\alpha), & 0 < \alpha \leqq \tfrac{1}{2}\pi\\
b_\pm &= (\pm\sin\beta, 0, \cos\beta), & 0 < \beta < \pi.
\end{aligned}
$$

The spherical arc length $\gamma = \pi(1-2/p)$ of P is thus given by

$$\cos\gamma = \langle a_+, b_+\rangle = \cos\alpha\cos\beta.$$

We now have two cases. If $p \geqq 5$, then $\cos\gamma < 0$, and hence $\cos\alpha > 0$, $\cos\beta < 0$. Then there is no plane H through v, such that all of $a_\pm, b_\pm$ lie in one of the open half-spaces bounded by H. If $p = 4$, then $\cos\gamma = 0$, and (since we can suppose $\alpha \leqq \beta$), we have $\cos\beta = 0$, and $\beta = \frac{1}{2}\pi$. Then $b_\pm$ and v are collinear, and so again no hyperplane H exists as above. We have now obtained our contradiction, since, if M were to exist, any vertex v of the 3-polytope conv$\,M$ would be a vertex of M possessing such a hyperplane H.

It is probable that one could similarly disprove the existence of any equiangular equivelar manifold of type $\{p,q\}$ with $p \geqq 4$ and $1/p + 1/q < \frac{1}{2}$, but the argument would necessarily be less simple than the one we have employed.

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